A note on the connection between product-form Jackson networks and counting lattice walks in the quarter plane

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1 Introduction

The ultimal goal is to find explicit solutions for counting generating functions (CGF) of some random walks in the quarter plane, starting from the wellknown product-form for the stationary distribution of customers in Jackson's stochastic networks. The first step is to solve a *relativety simple* boundary value problem (BVP) for the queueing system. Then, by using a variational principle based on the continuity with respect to adequate parameters, it is possible to transform some smooth curves in a continuous fashion and to get explicitly formulas for the conformal mappings of Riemann's theorem. This might be a new way of computing the gluing functions appearing in [1, 3], which are a key ingredient in the explicit formulas for CGFs.

2 An illustrative example of walks with steps East, South, North-West: tandem queues

We start by considering the related system consisting of two M/M/1 service stations in tandem, under Poisson (intensity λ) external arrival process and exponential service time distributions with equal service rates μ (this does not change the nature of the results).

Setting $\rho \stackrel{\text{def}}{=} \frac{\lambda}{\mu}$, the stationary distribution for the joint distribution of the number of units in the system satisfies the functional equation

$$Q(x,y)F(x,y) = \left(1-\frac{y}{x}\right)F(0,y) + \left(1-\frac{1}{y}\right)F(x,0), \tag{2.1}$$

with $Q(x,y) = \rho(1-x) + \left(1-\frac{y}{x}\right) + \left(1-\frac{1}{y}\right)$.

For $\rho < 1$, the solution of (2.1) is well-known to have the simple product-form

$$F(x,y) = \frac{(1-\rho)^2}{(1-\rho x)(1-\rho y)}. \tag{2.2}$$

Now, assume we do not know the simple expression (2.2). Then, according the method invented in [1], it is always possible to set a boundary value problem (BVP) of Riemann-Dirichlet-Carleman type for the function $F(0,y)$, which here has a simple formulation. We shall sketch the mains steps.

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As usual, we first restrict the equation (2.1) to the algebraic curve

$$\{(x,y) \in \mathbb{C}^2 : xyQ(x,y) = 0\}.$$

Solving $xyQ(x,y) = 0$ with respect to y yields

$$y^2 - x(\rho + 2 - \rho x)y + x = 0, \tag{2.3}$$

which has two roots $Y_1(x), Y_2(x)$ (the two branches of the global algebraic funtion $Y(x)$), such that

$$|Y_1(x)| \le 1 \leqslant |Y_2(x)|, \forall |x| = 1, \quad \text{and} \quad |Y_1(x)| \leqslant |Y_2(x)|, \forall x \in \mathbb{C}.$$

It is also worth noting

$$Y_1(x).Y_2(x) = x, \quad \forall x \in \mathbb{C}. \tag{2.4}$$

The discriminant of equation (2.3)

$$D(x) = x^2(\rho + 2 - \rho x)^2 - 4x = x(x-1)(\rho^2 x^2 - \rho(\rho+4)x + 4)$$

has four real zeros: $x_1 = 0 < x_2 = 1 < x_3 < x_4$, which are the branch points of the algebraic curve $Y(x)$ in the complex plane $\mathbb{C}_x$.

For $x \in [0,1]$, we have $Y_1(x) = \overline{Y_2(x)}$, since the two roots of $Q(x,y)$ considered as a polynomial in y are complex congugate, and $Y(x)$ describes a simple closed contour $\mathcal{L}$. Indeed $\mathcal{L}$ is the *smallest* component of a quartic curve, which in the $\mathbb{C}_y$-plane has the following equation, with $y = u + iv$,

$$\rho(u^2+v^2)^2 - (\rho+2)(u^2+v^2) + 2u = 0. \tag{2.5}$$

The contour $\mathcal{L}$ is situated entirely in the right half-$\mathbb{C}_y$-plane, inside the unit circle, and crosses the real axis at $u = 0$ and $u = 1$, where it has two vertical tangents. The interior domain bounded by $\mathcal{L}$ will be denoted by $\mathcal{L}^+$.

Letting x tend successively to the upper and lower edge of the slit (cut) $[0,1]$, we can eliminate $F(x,0)$ which is continuous on $[0,1]$. So, after a quick algebra, setting for short

$$\Phi(y) \stackrel{\text{def}}{=} F(0,y), \quad A(y) \stackrel{\text{def}}{=} \left(\frac{y}{1-y}\right)^2,$$

we can state the following BVP for $F(0,y)$.

Find a function Φ holomorphic in $\mathcal{L}^+$, the limiting values of which on the contour $\mathcal{L}$ are continuous and satisfy the relation

$$\boxed{A(t)\Phi(t) - A(\bar{t})\Phi(\bar{t}) = 0, \quad t \in \mathcal{L}}. \tag{2.6}$$

Remark 2.1. The problem (2.6) is a priori of generalized Dirichlet-Carleman type, *since the function $A(\cdot)$ has a zero at $t = 0$ and a pole at $t = 1$ on the contour*, but is otherwise analytic in $\mathcal{L}^+$. Therefore slightly sharper technicalities appear.

Setting

$$B(t) \stackrel{\text{def}}{=} \frac{A(\bar{t})}{A(t)} = \left(1 - \frac{1}{t}\right)^2 \Big/ \left(1 - \frac{1}{\bar{t}}\right)^2$$

the BVP (2.6) can be rewritten in the form

$$\boxed{\Phi(t) = B(t)\Phi(\bar{t}), \quad t \in \mathcal{L}}. \tag{2.7}$$

Checking that $B(t)$ is continuous on the contour $\mathcal{L}$, where it satisfies a Hölder condition, we get a classical Riemann-Hilbert problem, see [2].

In order to solve it, we introduce the conformal mapping $T(z)$ of the closed unit disc $\mathcal{D} = \{z : |z| \leq 1\}$ onto the region $\mathcal{L} \cup \mathcal{L}^+$. By symmetry with respect to the real axis in the respective planes, we can take

$$y = T(z), \quad \bar{y} = \overline{T(z)} = T(\bar{z}) = T(1/z), \quad \forall |z| \leqslant 1,$$

and we shall also choose $T(-1) = 0$, which implies necessarily $T(1) = 1$.

Introduce the new functions $\Psi(z) \stackrel{\text{def}}{=} \Phi(T(z)), C(z) \stackrel{\text{def}}{=} B(T(z))$, and denote by $\mathcal{U}$ the unit circle $|z| = 1$.

Then condition (2.7) can be put into Hilbert's standard form

$$\boxed{\Psi^+(t) = C(t)\Psi^-(t), \quad t \in \mathcal{U}}, \tag{2.8}$$

where $\Psi(\cdot)$ is sought to be sectionally analytic, of finite degree at infinity, and the solution in $\mathcal{L}^+$ is given by (see e.g. [2])

$$\Psi^+(z) = e^{\Gamma(z)} \quad \text{with} \quad \Gamma(z) = \frac{1}{2i\pi} \int_{\mathcal{U}} \frac{\log [t^{-\chi} C(t)] dt}{t - z}, \quad \forall |z| < 1. \tag{2.9}$$

where χ is the *index*, defined by

$$\chi = \mathcal{I}nd[C]_{\mathcal{U}} \stackrel{\text{def}}{=} \frac{1}{2\pi}[\arg C]_{\mathcal{U}} = \frac{1}{2i\pi}[\log C]_{\mathcal{U}}, \tag{2.10}$$

which represents the variation of the argument of $C(u)$, as u moves along the unit circle $\mathcal{U}$ in the positive direction. Hence, the function $\log [u^{-\chi} C(u)]$ is one-valued on $\mathcal{U}$.

From the expression of $B(t)$ given in (2.7), it appears that $|C(u)| = 1$ for $|u| = 1$, and $C(u)$ is also continuous at the points $u = 0, 1$. From these remarks, it is not difficult to prove that, for the special model under consideration, $\chi = 0$ [details are omitted].

Starting from the expression

$$C(t) = \left(1 - \frac{1}{T(t)}\right)^2 \Big/ \left(1 - \frac{1}{T(1/t)}\right)^2,$$

we can write $\log[C(t)]$ in the following form, which takes into account the singularities of the logarithms at $t = 1$ and $t = -1$ (respectively $y = 1$ and $y = -1$ in the original $\mathbb{C}_y$ plane).

$$\log[C(t)] = 2\left[\log \frac{1 - T(t)}{1 - t} - \log \frac{T(t)}{t + 1} - \log \frac{1 - T(1/t)}{1 - t} + \log \frac{T(1/t)}{t + 1}\right]. \tag{2.11}$$

Now the computation of $\Gamma(z)$ defined in (2.9) becomes straightforward by using the Cauchy residue theorem for the integrand (2.11). This yields

$$\Psi^+(z) = \left[\frac{(1 + z)(1 - T(z))}{(1 - z)T(z)}\right]^2, \quad \forall |z| \leq 1. \tag{2.12}$$

On the other hand, we have by (2.2)

$$F(0,T(z)) \equiv \Psi^+(z) = \frac{(1-\rho)^2}{1-\rho T(z)} \quad \forall |z| \le 1,$$

so that, comparing with (2.12), we see that $T(z)$ satisfies a polynomial equation of third degree. It is in fact simpler to write a closed form formula for the inverse mapping $S(y)$, which transforms conformally $\mathcal{L}_+$ onto the unit disc $|z| \leqslant 1$. We obtain

$$S(y) = \frac{(1-\rho)y - (1-\rho\sqrt{1-\rho y}}{(1-\rho)y + (1-\rho\sqrt{1-\rho y}}.$$

3 More general two-queue Jackson networks

In Fig. 3.1, we have dawn three generators representing the evolution of two-queue Jackson networks, which means that the invariant measure (assuming ergodicity conditions to be satisfied) has a product-form solution. From left to right, we refer to them as models $M1, M2, M3$. Then, in the context of Section 4 and according to the definition given in [1], the corresponding walks have respective *groups of order* $6, 6, 8$.

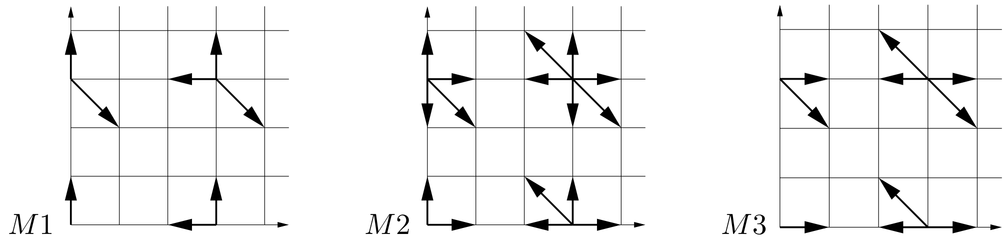

Fig. 3.1: Three examples of coupled queues representing a product-form Jackson network.

3.1 A BVP for the models of figure 3.1

Again we shall view $M2$ as a simple stochastic network of two coupled $M/M/1$ queues. The following choice for the rates of the generator of the underlying Markov process will be made, with obvious notations for the cardinal points, and $0 \le p_i \le 1,\ p_i + q_i = 1, i = 1, 2$.

$$\lambda_1 \text{ (E)}, \quad \lambda_2 \text{ (N)}, \quad p_1\mu \text{ (W)}, \quad p_2\mu \text{ (S)}, \quad q_1\mu \text{ (N-W)}, \quad q_2\mu \text{ (S-E)}.$$

Setting $\rho_i = \frac{\lambda_i}{\mu}, i = 1, 2$, it is not difficult to check that the invariant measure satisfies the following functional equation.

$$\begin{cases} R(x,y)G(x,y) = y\big[x - (p_1 + q_1 y)\big]G(0,y) + x\big[y - (p_2 + q_2 x)\big]G(x,0), \\ R(x,y) = xy\big[\rho_1(1-x) + \rho_2(1-y)\big] + y\big[x - (p_1 + q_1 y)\big] + x\big[y - (p_2 + q_2 x)\big]. \end{cases} \tag{3.1}$$

Then the solution of (3.1) turns out to have the nice product-form

$$G(x,y) = \frac{(1-r_1)(1-r_2)}{(1-r_1 x)(1-r_2 y)}, \tag{3.2}$$

where

$$r_1 = \frac{\lambda_1 + q_2\lambda_2}{(1-q_1q_2)\mu}, \quad r_2 = \frac{\lambda_2 + q_1\lambda_1}{(1-q_1q_2)\mu},$$

assuming the ergodicity conditions $r_i < 1, i = 1,2$ are satisfied.

Then, after setting for short $H(y) \stackrel{\text{def}}{=} G(0,y)$, we return to (3.1) to obtain the following BVP in the $\mathbb{C}_y$-plane (the contour $\mathcal{L}$ being, as before, the convenient component of a quartic curve), which is of the same type as (2.6) and obtained by eliminating $xG(x,0)$ in (3.1).

Find $H(\cdot)$ meromorphic in the domain $\mathcal{L}^+$ and satisfying

$$K(t)H(t) - K(\bar{t})H(\bar{t}) = 0, \quad t \in \mathcal{L}, \tag{3.3}$$

where

$$K(t) = \frac{t\big[X_1(t) - (p_1 + q_1 t)\big]}{\big[t - (p_2 + q_2 X_1(t))\big]},$$

noting that $H(\cdot)$ might have a pole inside $\mathcal{L}^+ \setminus \mathcal{D}$, and the function $X_1(\cdot)$ is the analogous of $Y_1(\cdot)$ defined in (2.4) for the kernel $R(x,y)$.

The next step is to use the above product form. Indeed, by (3.2), the right-hand side of (3.1) yields the following algebraic system

$$\begin{cases} \dfrac{y\big[x - (p_1 + q_1 y)\big]}{1 - r_2 y} + \dfrac{x\big[y - (p_2 + q_2 x)\big]}{1 - r_1 x} = 0, \\[2ex] \{(x,y) \in \mathbb{C}^2 : R(x,y) = 0\}, \quad |x|, |y| \leqslant 1. \end{cases} \tag{3.4}$$

We note first that the BVP (3.3) is equivalent to the BVP

$$(1 - r_2 t)H(t) - (1 - r_2 \bar{t})H(\bar{t}) = 0, \quad t \in \mathcal{L}, \tag{3.5}$$

which in the present situation admits (as expected!) precisely the unique solution $H(y) = \dfrac{H(0)}{1 - r_2 y}$.

To proceed as in Section 2 to find the conformal mapping $T(z)$ for the model $M2$, it is in fact necessary to solve *directly* the BVP (3.3). This is the matter of an ongoing work.

4 Application to counting walks in the quarter-plane

Let us recall below the basic assumptions.

- Walks start at the origin.
- The set $\mathcal{S}$ of admissible steps is included in the set of the 8 nearest neighbors, i.e., $\mathcal{S} \subset \{-1,0,1\}^2 \setminus \{(0,0)\}$. By using an extended Kronecker's delta, we shall write

$$\delta_{i,j} = \begin{cases} 1 & \text{if} \quad (i,j) \in \mathcal{S}, \\ 0 & \text{if} \quad (i,j) \notin \mathcal{S}. \end{cases}$$

Let $f(i,j,k) \stackrel{\text{def}}{=}$ the number of paths in $\mathbb{Z}_+^2$ starting from $(0,0)$ and ending at (i,j) after k steps. Then

$$F(x,y,s) \stackrel{\text{def}}{=} \sum_{i,j,k \geqslant 0} f(i,j,k) x^i y^j s^k \tag{4.1}$$

satisfies the functional equation

$$\boxed{K(x,y,s)F(x,y,s) = c(x)F(x,0,s) + \widetilde{c}(y)F(0,y,s) + c_0(x,y,s)}, \tag{4.2}$$

a priori valid in the domain $|x| \leqslant 1, |y| \leqslant 1, |s| < 1/|\mathcal{S}|$, where

$$\begin{cases} K(x,y;s) = xy\left[\displaystyle\sum_{(i,j)\in\mathcal{S}} x^i y^j - 1/s\right], \\ c(x) = \displaystyle\sum_{i\leqslant 1} \delta_{i,-1} x^{i+1}, \quad \widetilde{c}(y) = \displaystyle\sum_{j\leqslant 1} \delta_{-1,j} y^{j+1}, \\ c_0(x,y,s) = -\delta_{-1,-1} F(0,0,s) - xy/s, \end{cases}$$

As for the special model of Section 2, we have

$$K(x,y,s) = y^2 - x(1/s - x)y + x.$$

which we compare with the kernel given by (2.3), namely

$$xyQ(x,y) = y^2 - x(\rho + 2 - \rho x)y + x.$$

4.1 A varational principle for the curve $\mathcal{L}$ in the east, South and North-West model

The idea is to find a linear homeomorphism $\mathcal{H} : (x,y) \to (X,Y)$ *as simple as possible*, transforming $xyQ(x,y)$ into $K(X,Y,z) = 0$, for convenient values of the parameter ρ, so that z remains unchanged (note that we consider z as a parameter).

The change of variables $x = aX, y = bY$, with $ab \neq 0$, in (2.3) gives (after having divided by b^2)

$$Y^2 - \frac{aX(\rho + 2 - a\rho X)}{b} Y + \frac{aX}{b^2} = 0.$$

Hence, taking $a = b^2$, the identification of $\dfrac{a(\rho + 2 - a\rho X)}{b}$ with $\dfrac{1}{s} - X$ yields the two following necessary conditions:

$$b(\rho + 2) = \frac{1}{s}, \quad \text{and} \quad b^3\rho = 1.$$

Finally, we can completely specify the sought homeomorphism by

$$x = b^2 X, \quad y = bX, \quad \rho = b^{-3}, \quad s = \frac{b^2}{1 + 2b^3}. \tag{4.3}$$

Two facts should be quoted.

(i) Here b is real positive and, since $\rho < 1$, we have $b > 1$. Moreover, s given by (4.3) is smaller than $1/3$ and thus belongs to the initial domain of definition of (4.1).

(ii) In the sequel, s will not be necessarily real.

References

[1] Guy Fayolle, Roudolph Iasnogorodski, and Vadim A. Malyshev. *Random Walks in the Quarter Plane: Algebraic Methods, Boundary Value Problems, Applications to Queueing Systems and Analytic Combinatorics*, volume 40 of *Probability Theory and Stochastic Modelling*. Springer International Publishing, February 2017. *Second edition*, https://hal.inria.fr/hal-01651919.

[2] F.D. Gakhov. *Boundary value problems.* Pergamon Press, 1966.

[3] K. Raschel. Counting walks in a quadrant: a unified approach via boundary value problems. *Journal of the European Mathematical Society*, 14:749–777, 2012.